Homework 7, due 11/4

1. (a) Let $f, g : \Omega \to \mathbf{C}$ be holomorphic functions, and $\gamma : [0,1] \to \Omega$ a curve contractible to a point in Ω. Suppose that for all $t \in [0,1]$ we have $|g(\gamma(t))| < |f(\gamma(t))|$. Prove that

$$\sum_{a\in\Omega} n(\gamma, a)\mathrm{ord}_a f = \sum_{a\in\Omega} n(\gamma, a)\mathrm{ord}_a(f+g).$$

   (This result is called Rouché's Theorem.)

   (b) Show that the equation $z^4 + 26z + 2 = 0$ has exactly 3 distinct solutions in the annulus $5/2 < |z| < 3$. {*Hint: count solutions in the two disks* $|z| < 5/2$ *and* $|z| < 3$.}

2. Compute the integral

$$\int_0^\infty \frac{\ln x}{x^2 - 1}\, dx,$$

   using contour integration. {*Hint: you can consider a contour given by the boundary of the domain* $\{z : |z| < R, \mathrm{Im}\, z, \mathrm{Re}\, z > r\}$.}

3. Let $A \subset \mathbf{C}$ denote the half-disk $A = \{z : |z| < 1, \mathrm{Re}\, z > 0\}$, and B denote the quarter plane $B = \{z : \mathrm{Re}\, z, \mathrm{Im}\, z > 0\}$.

   (a) Find a biholomorphism $f : A \to B$. {*Hint: consider a fractional linear transformation mapping* i *to* 0*, and* $-i$ *to* ∞.}

   (b) Find a biholomorphism $g : B \to D(0,1)$ to the unit disk.

4. Suppose that $f : H \to \mathbf{C}$ is holomorphic, where H denotes the upper half plane. Suppose that $f(i) = 0$, and $|f(z)| < 1$ for all $z \in H$. How large can $|f'(i)|$ be?